Ecological Modelling 475 (2023) 110186

Contents lists available at ScienceDirect

Ecological Modelling

journal homepage: www.elsevier.com/locate/ecolmodel

Collective management of environmental commons with multiple usages: A guaranteed viability approach

Isabelle Alvarez [a,b,*], Laetitia Zaleski [c], Jean-Pierre Briot [c], Marta de A. Irving [d]

[a] UCA, INRAE, LISC, 9 avenue Blaise Pascal CS 20085, F-63178 Aubière, France
[b] CNRS, ISC-PIF, 113 rue Nationale, F-75013, Paris, France
[c] SU, CNRS, LIP6 4 place Jussieu, F-75005, Paris, France
[d] EICOS, IP, UFRJ Avenida Pasteur, 250 Rio de, Janeiro 22290-240, Brazil

ARTICLE INFO

Dataset link: https://forgemia.inra.fr/isabelle.alvarez/emlake

Keywords:
Viability theory
Guaranteed viability
Collaborative decision
Environmental commons
Lake eutrophication

ABSTRACT

In this paper we address the collective management of environmental commons with multiple usages in the framework of mathematical viability theory. We consider that stakeholders can derive from the study of their specific socioeconomic problem (i) the variables describing the different usage of the commons and its evolution (ii) and a representation of the desirable states for the commons. We then consider the guaranteed viability kernel, subset of the set of desirable states where it is possible to maintain the state of the commons even when its evolution is represented by several conflicting models. This approach is illustrated on a problem of lake eutrophication.

1. Introduction

Sustainable use of natural resources, environmental conservation, social inclusion and welfare, economic activity and development entail generally conflicting management objectives. In *The Tragedy of the Commons*, Hardin (1968) highlights the exhaustion of open-access resources by numerous users who hold similar views, but the same analysis can be made with different types of users whose activity is based on the resources. A lot of work on sustainability of natural resources is still focused on the allocation problem, where stakeholders are considered competitors for the share of quotas, for example, for the regulation of fisheries or water sharing (see references, for instance, in Parrachino et al. (2006) and Oubraham and Zaccour (2018)).

In order to take into account the interests of the different types of stakeholders, significant efforts have been made using the economics approach to assess the value of environmental and social services (see, for example, a framework in de Groot (2006)). When points of view are considered to be incommensurable, multi-criteria or viability theory approaches offer interesting alternatives. Even when stakeholders are considered to be competitors for one common resource, these approaches make it possible to take into account more indicators than the level of renewable resource and the profit directly based on it. For example, in a quantitative work on fishing regulation (Dowling et al., 2020), 21 score functions are designed for the regulation of fishing, depending on fish biomass and parameters computed each year according to the control scenario. Stakeholders' weighted preferences over the score functions are then optimized each year for different levels of the control variable. When stakeholders express different points of view, the mathematical viability theory (MVT) approach makes it possible to combine the different constraints placed on the system, without direct connection to the underlying profit of the related activity. For instance, in a hydro-power dam management problem (Alais et al., 2017), the main concern is maximizing the profit of the electricity provider with water control under uncertainty in water inflow and electricity price. Recreational and agricultural activities impose an additional seasonal constraint on the water level without further profit analysis. In Wei et al. (2013), the multi-objective concern of a tourist city is studied through the joint evolution of the number of tourists, tourism infrastructure and environment quality. The different stakes are represented by constraints on the level of these variables. The MVT algorithm identifies the area where it is possible to maintain the evolution of the three variables between these bounds. The potential of the MVT approach has been demonstrated in many other domains as described in the review by Oubraham and Zaccour (2018). In all these works, the model of the evolution of common resources or land uses, together with the impact of controls on the system (such as the total allowable catch

* Corresponding author at: UCA, INRAE, LISC, 9 avenue Blaise Pascal CS 20085, F-63178 Aubière, France.
E-mail addresses: Isabelle.Alvarez@inrae.fr (I. Alvarez), l.zaleski@ccr-zkr.org (L. Zaleski), Jean-Pierre.Briot@lip6.fr (J.-P. Briot), mirving@mandic.com.br (M. de A. Irving).

https://doi.org/10.1016/j.ecolmodel.2022.110186
Received 5 May 2022; Received in revised form 14 September 2022; Accepted 27 October 2022
Available online 24 November 2022
0304-3800/

in fishery regulations), is supposed to be consensual. It is generally taken from the literature or from previous work and parameters are calibrated from data and time series. In works cited in Oubraham and Zaccour (2018), the model representing the system at stake is always considered as consensual. In theoretical works, models can certainly be generic functions of variables, controls and uncertainties (as is the case in De Lara and Martinet (2009), Křivan (1991), Křivan and Colombo (1998) and Martinet et al. (2016)). But in their applications and in the other works cited in Oubraham and Zaccour (2018), when uncertainty is explicitly taken into account, it is in fact related to data and measurements used to assess parameters in the model, not to the definition of the model itself. As stated in Martinet et al. (2016), uncertainty affects parameters such as growth rate, recruitment or mortality in dynamic population models, unknown or unpredictable events such as climate fluctuations, or externalities such as price, as in games against nature. Models are supposed to be consensual with their explicit hypotheses (which are generally discussed). In Little et al. (2007) three models are considered for larval dispersal and modelers can parameterize the system to run simulations with their own choice of model. This is motivated by the possibility of studying different species, therefore for the viability study only one model is parameterized.

However, the ComMod approach (Etienne, 2014) has shown that modeling the evolution of the system at stake is difficult and hardly consensual, since scientific or technical viewpoints can be considered by stakeholders as viewpoints among others. The ComMod approach addresses this problem with serious games supported by simulation models (Barreteau et al., 2001) where stakeholders can test their hypotheses about the system evolution and the impact of actions. The process continues, with additional research if necessary, until a consensus on the model is reached. To take into account this discrepancy at the model level, we consider here that stakeholders have their own model of the evolution of the system with the impact of controls. We consider that stakeholders are able to define constraints on the key variables of their usages of the commons, such as the number of tourists, the quality of water (for example, measured in terms of concentration of pollutants or bacteria), the quality of the environment (for example, measured with biodiversity indicators related to the population of local species), etc. These constraints are generally seen as thresholds. We consider that the objective of the group of stakeholders is to define a set of states where the system can be maintained with appropriate controls. We use viability theory, in particular the concept of guaranteed viability set (Aubin, 1997), which is defined to take into account uncertainties (such as a move by nature, see, for instance, Bates and Saint-Pierre (2018)).

The paper is organized as follows: we first describe the problem and our hypotheses, together with a brief summary of MVT. We describe individual and group viewpoints as viability problems, and show why in this context of several models it is more difficult to seek a technically sound agreement. We illustrate this approach with a problem of lake eutrophication. In Section 3, we first present the perturbation embedding function, which allows us to consider each model as a perturbation of a central model. This formulation enables the definition of a guaranteed viability problem. In Section 4 we present and discuss the application to the management of the lake. We summarize the results and perspectives in the concluding section.

2. A viability-based framework

Here, we propose a framework for using MVT for collective management problems when stakeholders have their own view of the dynamics of the system.

2.1. Definition of the problem of collective management

2.1.1. Prototypical situation

Let us consider an entity or system $\mathcal{A}$ (for instance, a preserved area), in evolution, which is submitted to the management decision of a group of N stakeholders. We are interested here in collective management situations with the following characteristics:

(C1) All stakeholders are dealing with the same system, described by the same variables.

(C2) All stakeholders can have their personal view of the admissible controls which should be taken into account depending on the state of the system. However, we will only consider the control variables and values which are common to all so as to find a consensus solution.

(C3) All stakeholders have their own personal view of the evolution of the state of the system, including the impact of controls, which is represented by a controlled dynamical system as a set of differential inclusions (or difference equations).

(C4) All stakeholders can define a set of desirable states in which the state of the system must remain, so that they should be able to satisfy their own objectives.

(C5) All stakeholders have a management objective for the system: Maintaining its state in their own set of desirable states.

(C6) All stakeholders trust the same third party (for instance, an assistant software), and share with it their admissible control map (noted U_i), their set of desirable states (noted K_i) and their dynamics (noted S_i).

We consider here that the objective of the collective management of system $\mathcal{A}$ is to fulfill the objective of all stakeholders. It means maintaining the state of the system in all their desirable sets simultaneously.

In this situation and with this objective of collective management, finding a solution to this problem means finding a subset H of the state space of the system $\mathcal{A}$, and a control map $\tilde{U}$ defined on this subset, such that from any of its states, and for all stakeholders, every evolution governed by their own model S_i (with control selected from $\tilde{U}$) remains in their own set K_i forever. When the set H is not the empty set (and when the associated control map $\tilde{U}$ defines non empty control sets), they form a consensus solution in the sense that they make it possible to derive a satisfactory evolution of $\mathcal{A}$ for all parties.

Actually, if stakeholders' positions are too far apart, the solution will be the empty set. In particular, it will be the case if there are no common controls, or if the intersection of all desirable sets is the empty set. Thus in practice two more consensus conditions must be added:

(C7) Taking into account all the different stakeholders' sets of desirable states, it is possible to define a non-empty consensus set of desirable states, which is a subset of each of them.

(C8) Taking into account all the control maps of the different stakeholders, it is possible to define a consensus control map, defined for all the states of the consensus set of desirable states.

However, we do not require a consensual model.

2.1.2. Consideration of uncertainties and constraints

In this framework we address the problem of uncertainty in a worst-case approach (as is possible in the MVT approach). We want to find solutions which are still valid when stakeholders consider uncertainties in some parameter of their model: The proposed solutions will be valid for all the parameter values envisaged.

We also want to find solutions even when stakeholders have different views on the laws or processes that govern the dynamics. Comparing models, or developing a consensual model, can be a very time consuming and difficult task if data are scarce or imprecise, and if experiments are costly or difficult to perform. Here we focus on the management objective and propose solutions that are valid for all the models considered.

Regarding the set of desirable states, we are not interested in proposing a solution for an "average" stakeholder, which could be obtained by averaging the different constraints and, if needed, the different models. We are not looking for compromise solutions, but for a solution corresponding simultaneously to all the constraints expressed by the stakeholders.

We use as an illustration a problem of lake eutrophication as stated in Carpenter et al. (1999). Agricultural practices and other human activities can lead to lake pollution with phosphates. Phosphorus dynamics in the lake can lead to eutrophication, which negatively impacts the biodiversity of ecosystems, and represents a serious annoyance to residents and tourism activities. We consider that a committee is formed to study and manage the problem. It is composed of farmers and local elected authorities.

2.1.3. Notation and formalization

We consider a situation as described in the preceding subsection. An entity $\mathcal{A}$ is submitted to the management decision of a group of N stakeholders. We note $\mathcal{N} := \{1, \dots, N\}$. All conditions C1 to C8 are fulfilled.

We note $x \in \mathbb{R}^n$ the vector of state variables that describes the state of $\mathcal{A}$.

We note K the consensual set of desirable states. We have from C7:

$$K = \bigcap_{i \in \mathcal{N}} K_i \neq \emptyset.$$

From C8 we can consider the set-valued map of consensual admissible control $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ defined on K. It associates the state of $x \in K$ with $U(x) \neq \emptyset$, the set of controls admissible at state x.

Definition 1. (K, U) is the group project for $\mathcal{A}$.

From C3 we consider that all group members describe the dynamics of the state of $\mathcal{A}$ as a (possibly discrete) controlled dynamical system. We note $Sc(f, U)$ the continuous dynamical system defined by:

$$Sc(f,U) \begin{cases} x'(t) & = & f(x(t), u(t)) \\ u(t) & \in & U(x(t)) \subset \mathbb{R}^p \end{cases}. \tag{1}$$

where f is a function from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^n$ and U a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^p$. Similarly, we note $Sd(f, U)$ the discrete dynamical system defined by:

$$Sd(f,U) \begin{cases} x^{k+1} & = & f(x^k, u^k) \\ u^k & \in & U(x^k) \subset \mathbb{R}^p \end{cases}. \tag{2}$$

We note S_i the dynamical system that described the evolution of $\mathcal{A}$ for member i. We have $S_i = Sc(f_i, U)$ in the continuous case and $S_i = Sd(f_i, U)$ in the discrete case. The function $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ associates the variations of $\mathcal{A}$ state variables with the current values of the state and control variables, according to member i.

We consider here that the different stakeholders do not necessarily agree on dynamics, and that they are not compelled to make their belief public. But (from C6) they agree to share this information with a trusted third party.

In the lake and nearby farms problem, following Carpenter et al. (1999) and Martin (2004), we consider that all members agree that the key variables to the problem are the phosphorus input (noted L) and the phosphorus concentration in the lake (noted P). Everyone wants to keep the lake in an oligotrophic state, which supposes setting a concentration limit of phosphorus P_{max} in the lake (for example, established from previous observations). Everyone also wants to maintain or develop the agricultural activity, which supposes allowing a minimum amount of phosphorus input L_{min} in the lake. Thus, everyone agrees to maintain the state of the lake described by (L, P) in a set of desirable states $K = [L_{min}, +\infty) \times [0, P_{max}]$. We consider that the committee agrees to the possibility of controlling the rate of variation of the phosphorus input and to maintain this rate between boundaries, hence $U = [u_{min}, u_{max}]$. This can be done by farmers controlling their fertilizer input, by the greater or lesser use of wetlands or by the use of water treatment plants (Gajardo et al., 2017).

We suppose in this simple illustration that the models of the different stakeholders differ only in the value of some parameters. The dynamics for member i are defined according to Carpenter et al. (1999) and Martin (2004) with the constraints on L_{min} and P_{max}:

$$S_i = S(b_i, r_i, q_i, m_i) \begin{cases} \frac{dL}{dt} & = & u \in U = [u_{min}, u_{max}] \\ \frac{dP}{dt} & = & -b_i P(t) + L(t) + r_i \frac{P(t)^{q_i}}{m_i^{q_i} + P(t)^{q_i}} \end{cases} \tag{3}$$

where b_i is the rate of loss (due to sedimentation and outflow), r_i, q_i and m_i are parameters of the sigmoid-like (s-shaped) dynamics of phosphorus recycling in the lake, which are generally set by calibration from observations: r_i is the maximum rate of recycled phosphorus, m_i is the concentration of phosphorus at which the recycling rate is half its maximum and q_i is a parameter of the steepness of the dynamics (see Carpenter et al. (1999) for more details).

The objective of all stakeholders is to maintain the state of $\mathcal{A}$ in K, according to their own dynamical model S_i. The objective of the group is to fulfill all these objectives simultaneously.

2.2. Brief summary of mathematical viability theory

Referring to Aubin (1991), we define viable evolutions and the viability kernel. We note $S = S(f, U)$ the system $Sc(f, U)$ (1) (resp. $Sd(f, U)$ (2) in the discrete case).

Definition 2. An evolution of the system S is viable in K if and only if its trajectory remains in K. In the continuous case: $\forall t \in \mathbb{R}^+ \; x(t) \in K$. In the discrete case: $\forall k \in \mathbb{N} \; x^k \in K$.

Definition 3. A set L is viable for the system S if for all $x \in L$ there is an evolution of S starting at x and viable in L.

Definition 4. The viability kernel associated with system S under constraint K is the set of all states in K from which there is an evolution of S starting at x and viable in K.

Under some general conditions listed in Appendix A, the viability kernel is a closed set. In the interior of the viability kernel, all controls are viable, hence viable controls on the boundary show how it is possible to maintain the system in the constraint set. This information can be used to define control strategies.

Definition 5. A control map with images restricted to viable controls only is called a "viable regulation map".

Proposition 1 (*Aubin (1991)*)**.** *If L is a viable set for the system $S(f, U)$, let $\tilde{U}$ be a viable regulation map, then L is a viable set for the system $S(f, \tilde{U})$. Moreover, for all $x \in L$, any evolution starting from x and governed by $S(f, \tilde{U})$ is viable in L. L is called an "invariant set" for dynamics $S(f, \tilde{U})$.*

From any state in the viability kernel, it is always possible to find a control function that allows the state of the system to stay in the viability kernel indefinitely. Conversely, from any initial state outside the viability kernel, there is no way to prevent the exit in finite time of an evolution governed by system S.

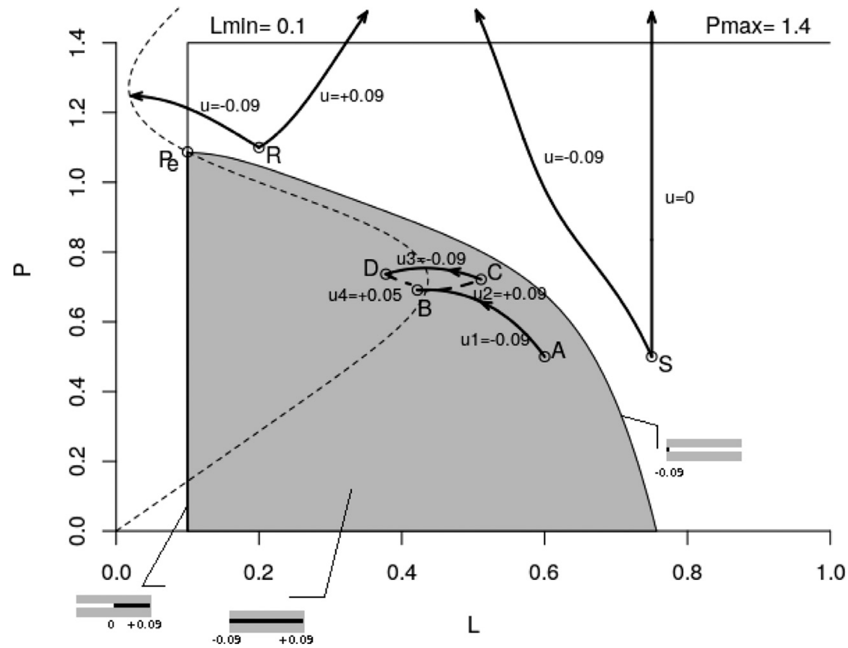


Fig. 1. Viability kernel (in gray) of the lake and neighboring farms problem, with $L_{min} = 0.1$, $P_{max} = 1.4$ (L and P in μg L^{-1}), $U = [-0.9, 0.9]$, dynamics parameters $q = 8$, $m = r = 1$, $b = 0.7$. Constraint set boundary is in solid black lines ($L = L_{min}$, $P = P_{max}$). The curve of equilibria is dashed. Viable controls are shown as a black line in cartouches. A viable trajectory starting from A is shown ($u = -0.09$ from A to B, then a cycle with $u = +0.09$ from B to C, $u = -0.09$ from C to D, $u = +0.05$ from D to B). From S where the lake is still in an oligotrophic state, even with maximum effort from the farmers ($u = u_{min}$) the concentration of phosphorus becomes too high. From state R also outside the viability kernel, all trajectories leave the constraint set, leading the lake to an eutrophic state or farmers' activity to an unsustainable state.

In the case of the lake and its neighboring farms, it is shown in Martin (2004) that the viability kernel associated with system (3) submitted to the constraint $(L, P) \in K = [L_{min}, +\infty) \times [0, P_{max}]$ is not empty when P_{max} is greater than the smallest P-value of the equilibria associated with L_{min} (an equilibrium P-value is defined by $\frac{dP}{dt} = 0$). For example, in Fig. 1, the state (L_{min}, P_e) is an equilibrium with $P_e \leq P_{max}$, and thus the viability kernel is not empty. When the curve of Equilibria intersects the half-line $(L \geq L_{min}, P = P_{max})$ at (L_e, P_{max}), the boundary of the viability kernel is delimited by the segment line $(L = L_{min}, P \leq P_{max})$, the segment line $(L_{min} \leq L \leq L_e, P = P_{max})$ and the integral curve of the dynamics with control $u = u_{min}$ arriving in (L_e, P_{max}). When the equilibrium curve do not intersect the half-line $(L \geq L_{min}, P = P_{max})$, as in Fig. 1, we note P_e the P-value of the highest equilibrium on the segment line $(L = L_{min}, P \leq P_{max})$. In that case, the boundary of the viability kernel is delimited by the segment line $(L = L_{min}, P \leq P_e)$ and the integral curve of the dynamics with control $u = u_{min}$ passing through (L_{min}, P_e).

Fig. 1 shows the viability kernel for the lake and neighboring farms problem in this latter case, for a given set of parameters for system (3) and constraint set K. From any state in this viability kernel, it is possible to find a trajectory that stays in the viability kernel indefinitely. Fig. 1 presents an example of a viable trajectory from a state in the viability kernel. It also shows examples of states outside the viability kernel; even the most severe control cannot prevent trajectories from leaving the constraint set. Either the lake will shift to an eutrophic state, or the economic activity will be jeopardized.

From a state outside the viability kernel, every evolution governed by system (3) with this particular set of parameters, choice of constraint set and control interval will exit the constraint set. In general, dealing with states outside the viability kernel entails studying the resilience (as in Martin (2004)) or redefining the problem. This can be done by relaxing the constraints on the desirable set (when it is possible), by allowing more efficient controls which are not presently part of the admissible controls, or by modifying the dynamics. This last option is generally more difficult to implement, since it involves modifying the lake itself (see Liu et al. (2015) for example of such actions).

2.3. Viewpoints as viability problems

Objectives. Let (K, U) be the group project. We recall the objectives stated in Section 2.1: The objective of all stakeholders is to maintain the state of $\mathcal{A}$ in K, according to their own dynamical model $S_i = S(f_i, U)$. The objective of the group is to fulfill all these objectives simultaneously.

Fig. 2 summarizes the implications of considering different usages and stakeholders for the lake and nearby farms problem. All stakeholders can work on a solution to this project according to the dynamics they assume for $\mathcal{A}$. Then the group can work on a solution from all members' solutions. Although the intuition is to work from the set of individual solutions, in this section we show that this approach is difficult to implement.

2.3.1. Individual viewpoint

Let $L_i \subset K$ be a non-empty viable set for system $S_i = S(f_i, U)$ submitted to constraints K. Then from all states in L_i there is at least one viable evolution governed by S_i that stays in L_i. From the viewpoint of member i, L_i is a solution state set to the management of $\mathcal{A}$.

Definition 6. $L_i \subset K$ is a solution state set for member i for project (K, U) if L_i is a non-empty viable set for member i's dynamics.

We note $\mathrm{viab}_i(K)$ the viability kernel associated with member i's project with dynamical system S_i submitted to the viability constraint K.

In the case of the lake and its neighboring farms, Fig. 1 shows the viability kernel for the dynamics (3) submitted to constraint set $K = [L_{min}, +\infty) \times [0, P_{max}]$ for the particular values of the dynamics parameters (noted as farmers' representative in Fig. 3).

2.3.2. Group viewpoint

In the following we suppose that all group members can propose their own individual solution to the management of $\mathcal{A}$:

$\forall i \in \mathcal{N}, \mathrm{viab}_i(K) \neq \emptyset$

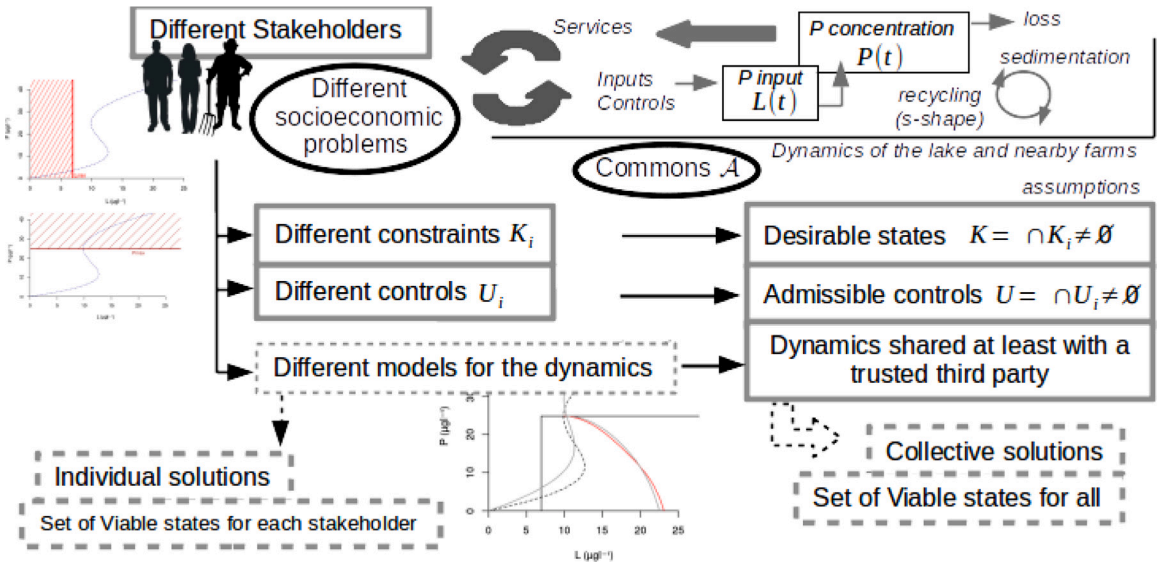


Fig. 2. Diagram of the finding of management solutions for the lake and nearby farms (system $\mathcal{A}$) in the framework of viability theory with different stakeholders. Gray arrows denote relationship in the dynamical model. Large gray arrows represent interaction with stakeholders' models and dynamics (which are not explicit). Black arrows represent the viability analysis process. Dotted lines and arrows show the main focus of the article.

We note $H = \bigcap_{i \in \mathcal{N}} \text{viab}_i(K)$. If $H = \emptyset$, a negotiation should obviously take place between stakeholders, since there is no way to operate $\mathcal{A}$ and satisfy the group members. When the intersection is not empty, it seems a good candidate. For example, the intersection of viability kernels has already been proposed as a solution to ensure the viability of two fishing fleets operating on the same resource. In Sanogo et al. (2012), the intersection of the viability kernel of both fleets is viable for each fleet if they change their effort at the same time when necessary, which supposes a high level of cooperation. But unfortunately, this is not always the case.

Proposition 2. *The intersection of all members' viability kernels is not necessary viable for all members.*

Proof. The problem of the lake and neighboring farms gives a counter-example. We consider here two stakeholders, say, a mayor and a farmer's union representative (respectively noted with m and f indices). Both stakeholders interpret the observations in different ways, thus they adopt different values for the parameters of dynamics (3). Their respective viability kernel (viab_m and viab_f) associated with the constraint set $K = \left[L_{min}, +\infty\right) \times \left[0, P_{max}\right]$ is shown in Fig. 3. For the particular parameters chosen, the intersection H is not empty. For $x = (L, P) \in viab_i$, viable controls are defined by $\tilde{U}_i$ with $\tilde{U}_i(L_{min}, P) = [0, u_{max}]$, $\tilde{U}_i(L, P) = \{u_{min}\}$ when (L, P) is on the boundary of viab_i with $L \neq L_{min}$, and otherwise $\tilde{U}_i(x) = U$. Nevertheless, state A in the intersection is not viable for the mayor. State A is on the boundary of the mayor's viability kernel; thus $\tilde{U}_m(A) = \{u_{min}\}$ and the only viable control for the mayor is $u = -0.09$. But the trajectory starting at A and governed by (S_m) with $u = -0.09$ stays on the boundary of viab_m and therefore it leaves H. ■

Definition 7. Let $L \subset K$ and let $(U_i)_{i \in \mathcal{N}}$ be control maps defined on L. Let U be defined for all $x \in L$ by $U(x) = \bigcap_{i \in \mathcal{N}} U_i(x)$. U is called "the intersection of $(U_i)_{i \in \mathcal{N}}$ on L".

We note $\tilde{U}$ the regulation map defined on the intersection H of all viability kernels of the group members by the intersection of all the corresponding viable regulation maps: $\tilde{U}(x) = \bigcap_{i \in \mathcal{N}} \tilde{U}_i(x)$. Obviously, if there is a state $z \in H$ such that $\tilde{U}(z) = \emptyset$, it means that members cannot agree on a way to control the evolution of $\mathcal{A}$ for this particular state. Unfortunately, even if all members agree on controls on H, it is not sufficient to reach a consensus.

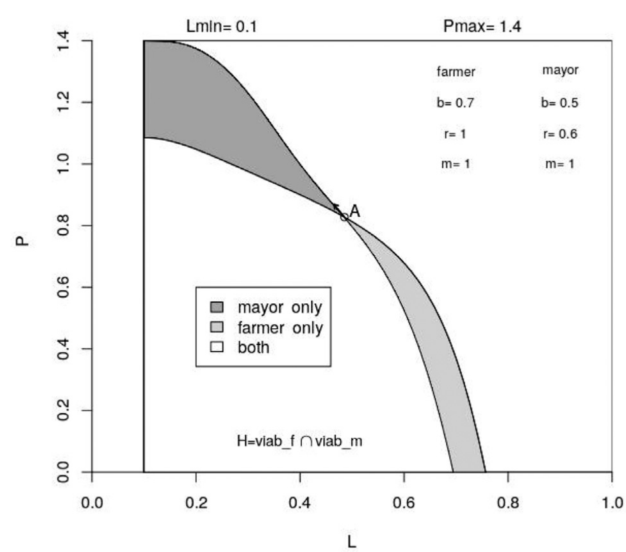


Fig. 3. Viability kernels of two stakeholders in the lake and neighboring farms problem, with $L_{min} = 0.1$ and $P_{max} = 1.4$ (L and P in μg L^{-1}), $U = [-0.9, 0.9]$, shared parameters value $q = 8, m = 1$. In white, the intersection H of the viability kernels. In dark (resp. light) gray, the complementary area of the mayor (resp. farmer) viability kernel. State A is not viable in the intersection for the mayor. The arrow shows the trajectory of state A according to the mayor: it leaves the white area.

Corollary 1. *Let $\tilde{U}$ be the intersection on $H = \bigcap_{i \in \mathcal{N}} \text{viab}_i(K)$ of the viable regulation map U_i on each $\text{viab}_i(K)$ of each member $i \in \mathcal{N}$. $Dom(\tilde{U}) = H$ is not a sufficient condition for H to be a viable set for all members.*

Proof. In the previous example, we can derive that $\tilde{U}_m(L, P) = \tilde{U}_f(L, P)$ for all (L, P) in the intersection except on the set H_b of the boundary of H wherever $L \neq L_{min}$. On the part of the boundary of H which is the boundary of $viab_m$ only, $\tilde{U}_m(L, P) = \{u_{min}\}$, while $\tilde{U}_f(L, P) = U$ (and conversely on the boundary of $viab_f$ only). Thus, for $(L, P) \in H_b$, $\tilde{U}(L, P) = \{u_{min}\}$ so $Dom(\tilde{U}) = H$. Nevertheless, state A is not viable in H for the mayor. ■

From Proposition 2 and Corollary 1 we propose the following definition for a technically sound consensus solution to the management of $\mathcal{A}$.

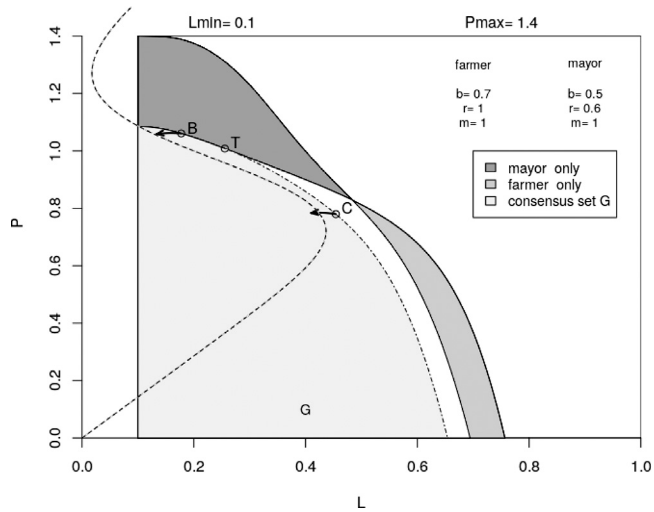


Fig. 4. A consensus state set for the two stakeholders in the lake and neighboring farms problem with parameters from Fig. 3. In very light gray, the consensus state set G, delimited in dash-dotted line by the trajectory following the mayor's dynamics that is tangent to the boundary of the viability kernel of the farmer at state T (solid line). From each state (L, P) of this trajectory before the tangent state T (with $L > L_T$), the evolution governed by the farmer's system starting at these states with $u = u_{min}$ leaves the boundary to evolve inside G, as is shown for state C. Correspondingly, from each state of the boundary of the viability kernel of the farmer with $L_{min} < L < L_T$, the evolution governed by the mayor's system starting at these states with $u = u_{min}$ leaves the boundary to evolve inside G, as it is shown for state B. G is a guaranteed viability kernel for both dynamics. The dashed line represents the line of equilibrium for the farmers' representative.

Definition 8. Let (K, U) be the project of the management group for $\mathcal{A}$. A set of state $H \subset K$ is a consensus solution if H is a viable set for each member and if the domain of the intersection $\tilde{U}$ of the viable regulation maps of each member on H, $(U_i)_{i\in\mathcal{N}}$, is such that $Dom(\tilde{U}) = H$.

In fact, to be a consensus solution, a subset H of the constraint set has to be viable for all members and each viable state has to share at least one viable control for all members. In that case it is possible for the group member to reach an agreement on the control, regardless of trajectories. For example, in the discrete case, from any state x_0 of H, all group members share at least one viable control value that allows the state of $\mathcal{A}$ to stay in H. Since the dynamics they consider are different, there is generally no consensus on state x_1. But as long as the group members still share a viable control value, they can still agree on it. When this is no longer the case, for example, at step n, the true value of the state of $\mathcal{A}$ can be measured to continue this process from x_n as a new starting point. In the continuous case, when H is a closed set, such situations arise only on the boundary of H.

Fig. 4 shows a consensus state set for the lake and neighboring farms system (3) with the parameters of Fig. 3. The associated regulation map $\tilde{U}$ is such that $\tilde{U}(L_{min}, P) = [0, u_{max}]$, $\tilde{U}(L, P) = \{u_{min}\}$ when (L, P) is on the boundary with $L \neq L_{min}$, and otherwise $\tilde{U}(L, P) = [u_{min}, u_{max}]$. For every state (L_0, P_0) of the consensus state set G, there is an evolution governed by system (3) for each stakeholder, starting at (L_0, P_0), with the same $u(0) \in \tilde{U}(L_0, P_0)$ that stays in G. In the interior of the intersection of the viability kernels, this property is also verified since for every state in the interior, all controls are viable in the viability kernel of each stakeholder. For states on the boundary with $L = L_{min}$, for both the mayor and the farmers' representative, several evolutions are viable, in particular with $u = 0$. The consensus state space is delimited by the boundary of the farmers' representative and the trajectory governed by the mayor's dynamics with the minimum control value that stays in the viability kernel of the farmers' representative with largest L when $P = 0$.

In the case of the lake and neighboring farms problem, with only two group members, it is possible to define a consensus state set because of the properties of the dynamics, for which the line of equilibrium is known, and the viability kernels and the trajectories corresponding to minimum control value u_{min} can be easily defined and computed (Martin, 2004). For more general cases it is necessary to propose a method that can be applied without such knowledge. We present such a method in the next section.

3. Consensus with guaranteed viability

3.1. Embedding function for the dynamics

Since the group members have their own definition for the dynamics, all members can see others' definitions as perturbations of their own. We show here that is possible to define the dynamics of $\mathcal{A}$ embedding all members' definitions seen as perturbations. The dynamics of $\mathcal{A}$ depends on the state of $\mathcal{A}$, $x(t)$, on the control chosen in $U(x(t))$ and on perturbations occurring from a set $V(x(t)) \subset \mathbb{R}^q$ that depends on the state of $\mathcal{A}$. In the continuous case we have:

$$Svc(f, U, V)\begin{cases} x'(t) & = & f(x(t), u(t), v(t)) \\ u(t) & \in & U(x(t)) \\ v(t) & \in & V(x(t)) \end{cases} \tag{4}$$

In the discrete case:

$$Svd(f, U, V)\begin{cases} x^{k+1} & = & f(x^k, u^k, v^k) \\ u^k & \in & U(x^k) \subset \mathbb{R}^p \\ v^k & \in & V(x^k) \subset \mathbb{R}^q\ , \end{cases} \tag{5}$$

where f associates the new state of $\mathcal{A}$ with its present state, a control chosen in $U(x(t))$ and a perturbation in $V(x(t))$. $Svc(f, U, V)$ and $Svd(f, U, V)$ are called "dynamical controlled tychastic systems" (Aubin, 1997).

Definition 9. We say that system $Svc(f, U, V)$ (4) (resp. $Svd(f, U, V)$ (5)) embeds system $Sc(f_i, U)$ (1) (resp. $Sd(f_i, U)$ (2)) for $i \in \mathcal{N}$, and call the corresponding pair (f, V) an embedding solution if and only if:

$$\forall x \in K, \forall u \in U(x), \forall i \in \mathcal{N}, \exists v_{i,u,x} \in V(x), f_i(x, u) = f(x, u, v_{i,u,x}) \tag{6}$$

We show in Appendix B that under some general conditions a system (4) (resp. (5) in the discrete case) can embed $Sc(f_i, U)$ (1) (resp. $Sd(f_i, U)$ (2)) for all $i \in \mathcal{N}$.

For example, for the problem of the lake and neighboring farms, the dynamics for every group member are defined from S_{lake} in Eq. (3) by $f_i : \mathbb{R}^2 \times \mathbb{R} \rightsquigarrow \mathbb{R}^2$, with:

$$f_i((x_1, x_2), u) = \begin{pmatrix} u \\ -b_i x_2 + x_1 + r_i \frac{x_2^q}{m + x_2^q} \end{pmatrix} \tag{7}$$

where parameters m and q have consensus values among the group, whereas parameters b_i and r_i have not. Then, by defining $V = [\min_{i\in\mathcal{N}}(b_i), \max_{i\in\mathcal{N}}(b_i)] \times [\min_{i\in\mathcal{N}}(r_i), \max_{i\in\mathcal{N}}(r_i)]$ and f as:

$$f((x_1, x_2), u, v) = \begin{pmatrix} u \\ -v_1 x_2 + x_1 + v_2 \frac{x_2^q}{m + x_2^q} \end{pmatrix} \tag{8}$$

with $v = (v_1, v_2) \in V$, Eq. (6) is verified, since in this simple case we have:

$$\forall x \in K, \forall u \in U(x), \forall i \in \mathcal{N}, v_{i,u,x} = (b_i, r_i).$$

In the following, we assume that the trusted third party has defined a map f and a perturbation map V such that system (4) (resp. (5) in the discrete case) describes the dynamics of $\mathcal{A}$, embedding the viewpoint of all group members.

The objective is then to find a consensus solution for $\mathcal{A}$ which will guarantee the viability for each member with shared viable controls.

3.2. Guaranteed viability with embedding dynamics

We recall here some definitions and properties of the mathematical theory of viability, from Aubin (1991) and Lavallée (2020), relative to guaranteed viability.

Definition 10. A solution $x(.)$ of system (4) (resp. (5)) is an evolution $(t \mapsto x(t))$ (resp. $(x^k)_{k\in\mathbb{N}}$) such that there is a control function $(t \mapsto u(t))$ (resp. $(u^k)_{k\in\mathbb{N}}$) and a perturbation function $(t \mapsto v(t))$ (resp. $(v^k)_{k\in\mathbb{N}}$) such that system (4) (resp. (5)) is verified for almost all $t \geq \mathbb{R}^+$ (resp. for all $k \in \mathbb{N}$).

Definition 11. An evolution $x(.)$ (resp. (x^k)) solution of system (4) (resp. (5)) is viable in L if and only if its trajectory remains in L.

Following Aubin (1991), Doyen (2000) and Lavallée (2020), we recall the property of guaranteed viability.

Definition 12 (*From Aubin (1997)*)**.** A set L verifies the property of guaranteed viability for $Svc(f,U,V)$ (4) (resp. $Svd(f,U,V)$ (5)) if there is a regulation map $\tilde{U}$ defined on L with a non-empty subset of U images, i.e., $\forall x \in L, \tilde{U}(x) \neq \emptyset$ and $\tilde{U}(x) \subset U(x)$ such that for all x_0 in L, all evolutions starting at x_0 and governed by $Svc(f,\tilde{U},V)$ (resp. $Svd(f,\tilde{U},V)$) are viable in L.

Definition 13. The guaranteed viability kernel associated with a set K is the largest set in K with the property of guaranteed viability (for λ Lipschitz controls in the continuous case, see Appendix A for definition).

We have seen that the intersection of each member's solution is not necessarily a solution for all members. We show now that the guaranteed viability kernel is a consensus solution (according to Definition 8).

Let $Svc(f,U,V)$ (resp. $Svd(f,U,V)$) be an embedding solution for all group members, which fulfilled conditions of Proposition 5. Let $\mathrm{Guar}_K \neq \emptyset$ be the guaranteed viability kernel for system $Svc(f,U,V)$ with λ Lipschitz constant (resp. $Svd(f,U,V)$) associated with constraint set K. Then we have the following property:

Theorem 1. *The guaranteed viability kernel associated with K for $Svc(f,U,V)$ (with λ Lipschitz constant in the continuous case) (resp. $Svd(f,U,V)$ in the discrete case) is a consensus solution to the management of $\mathcal{A}$.*

The demonstration can be found in Appendix A.3. We note $\tilde{U}$ the viable regulation map associated with Guar_K. The basic idea is that in Guar_K, from member i's perspective, an evolution governed by system $S_i(f_i,\tilde{U})$ is also governed by the embedding system $Svc(f,U,V)$ (resp. $Svd(f,U,V)$), thus it remains in Guar_K; Therefore Guar_K is a viable set for each member i. ■

Under some general conditions, the guaranteed viability kernel is a closed set (see Proposition 5 in Appendix A.3) and it is possible to retrieve the value of guaranteed viable controls on its boundary.

4. Application to the problem of lake eutrophication

4.1. Lake bourget case

We consider Lake Bourget, which is the biggest lake located entirely within France. It is monitored by the inter-syndicate committee CISALP, which is in charge of the design, monitoring and management of contractual actions for depollution and restoration of Lake Bourget. The lake had experienced a long eutrophication period: in 1974, the incoming amount of phosphorus in the lake was approx. 300 tons per year, and in 1989 the in-lake concentration was above 150 mg m^{-3} (Vinçon-Leite and Tassin, 1990), while OECD norms assess the in-lake concentration to be a maximum of $P_o = 10$ mg m^{-3} (equivalent to 36 tons) for the oligotrophic state (from Vollenweider (1982)). Similarly a threshold for a mesotrophic state of $P_m = 35$ mg m^{-3} can be defined. At concentrations above P_m the lake is assumed to be in an eutrophic state. Since the lake is monitored and the data are available, it is possible to calibrate the equations of system (3) for Lake Bourget. Calibration coefficients from Brias et al. (2018) are given in Table 1.

Table 1
Parameters of the Lake Bourget model. Unit conversion from tons to mg m^{-3} of state values, and parameters r and m, is done by dividing by the volume v of the lake in billions of m^3. We have $v = 3.6\ 10^9$ m^3 (and it is assumed to be constant).

Parameter	b	r	q	m
Value	State unit in tons			
	2.2676	367.04	2.222	96.85
	State unit in mg m^{-3} (or μg l^{-1})			
	2.2676	101.96	2.222	26.90

Lake Bourget offers multiple services apart from being a freshwater reserve. It is an area of major ecological interest for its flora and fauna as well as the diversity of its biotopes. Several areas of the lake are classified as protected areas. It supports extensive tourism and recreational activities (water-sport, fishing, beaches, marinas) and cultural activities linked in particular to historical heritage and literature. Several other services are being considered, such as the production of hydrothermal energy. Although the state of the lake has considerably improved, it is still considered as oligo-mesotrophic. Its dynamics can be unstable due to P loading and several blooms of cyanobacteria have been recently observed.

Agriculture is now the main source of P loading since major prevention measures have been taken since 1980. In particular the effluents of water treatment plants are no longer discharged into the lake. The control of incoming P is considered as essential because of the potential lagged impact of P release from sediments (Jacquet, 2018).

We consider a scenario where the CISALP in its mission of negotiation would approach agricultural unions, local representatives and managers of tourism activities, to form a committee for the control of P loading in the lake to prevent eutrophication and its consequences. The effort in P loading we consider is a limit on its rate of variation as is common in environmental actions. The control can be implemented by changes in agricultural practice and by the use of wetlands or retention basins. Currently, retention basins are being built to regulate the inflow of polluted water.

We consider that the committee members agree on the state variables and are aware of models for Lake Bourget (such as Brias et al. (2018)), but they disagree on the value of parameters or even on the model formulation. We consider that they agree on a set K of desirable states and on a set of admissible controls U.

4.2. Scenario and results

We consider a scenario where members of the committee agree on the possibility of controlling the rate of phosphorus loading (L). They consider different parameters sets for model (3), and a different formulation for the process of recycling from sediments. Some members consider that the recycling process can have a greater effect at low values of total P than with model (3). A different formula for the sigmoid-like function is used in that case, as shown in Eq. (9), with parameter λ_i controlling the S-shape. For a small value of $\lambda_i > 0$ the recycling occurs also for low levels of in-lake P, hence the lower branch of the equilibrium curve is actually higher.

$$S_i' \begin{cases} \frac{dL}{dt} &= u \in U = [u_{min}, u_{max}] \\ \frac{dP}{dt} &= -b_i P(t) + L(t) + r_i \frac{P(t)}{P(t)+m_i e^{(-\lambda_i (P(t)-m_i))}} \end{cases} \tag{9}$$

Table 2
Beliefs regarding the model and parameters of Lake Bourget dynamics. Member 1 trusts the literature. Member 2 considers the same model with a range of values for the steepness, as does member 3 for another parameter. Member 4 considers the alternative model with a range of values for its steepness.

Parameters b, r, m in μg l^{-1}	b_i P loss	r_i max. rate	m_i P value at half max. rate	α_i model type	q_i steepness model (3) S_i	λ_i steepness model (9) S'_i
Table 1 /Member 1	2.2676	101.96	26.90	1	2.222	–
Member 2	2.2676	101.96	26.90	1	[2.2, 2.3]	–
Member 3	[2.2, 2.3]	101.96	26.90	1	2.222	–
Member 4	2.2676	101.96	26.90	0	–	[1/19, 1/16]

It is possible to embed both model types by considering an additional parameter $\alpha_i \in [0,1]$ which controls the predominance of one type over the other. The corresponding model is represented in Eq. (10).

$$B_i \begin{cases} \frac{dL}{dt} = u \in U = \left[u_{min}, u_{max}\right] \\ \frac{dP}{dt} = -b_i P(t) + L(t) + (1-\alpha_i) r_i \frac{P(t)^{q_i}}{m_i^{q_i} + P(t)^{q_i}} + \alpha_i r_i \frac{P(t)}{P(t) + m_i e^{(-\lambda_i (P(t) - m_i))}} \end{cases} \tag{10}$$

The committee members' beliefs that we consider are summarized in Table 2. Regarding the definition of the constraint set, we consider that agricultural activity leads to at least 25 tons of incoming P each year. This value is arbitrary but it is lower than the mean loading between 2004 and 2016, which was above 33 tons/year (see Brias et al. (2018)). Thus we choose as lower limit $L_{min} = 25/v \approx 6.94$ μg l^{-1}. Considering the desirable threshold for in-lake P, we consider an optimistic scenario with the value of the mesotrophic equilibrium as maximum, $P_{max} = 24.76$ μg l^{-1}. The constraint set for this scenario is $K = \left\{(L,P), L \geq L_{min}, P \leq P_{max}\right\}$. As admissible controls, we consider that the maximum rate for the reduction of incoming P is half the maximum difference Δ of loading between two consecutive years between 2004 and 2016. For the increase of the loading we consider that the maximum rate can be Δ. Thus the set of admissible control is $U = [-\frac{\Delta}{2}, \Delta]$, with $\Delta \approx 3.15$ μg l^{-1} y^{-1}. When parameters are in a range, we consider the embedding dynamics Sv as in Eq. (8) with the corresponding parameter as v and V its range. For each member i, it is possible to define for project (K,U) a viability problem either as in Section 2.3.1, when parameters have fixed values, or a guaranteed viability problem as in Section 3.2, when parameters are in a range. Fig. 5 shows member 1's viability kernel and guaranteed viability kernels computed for members 2 to 4.

Applying the method described in the previous section, we define an embedding function f_B for the group from model (9) and Table 2. We then define the guaranteed viability problem Bv (Eq. (13)) associated with f_B.

$$f_B(x,u,v) = \begin{pmatrix} u \\ -v_1 P + L + r\left((1-v_2)\frac{P^{v_3}}{m^{v_3}+P^{v_3}} + v_2 \frac{P}{P + m e^{(-v_4(P-m))}}\right) \end{pmatrix} \tag{11}$$

where v_1 stands for parameter b_i, v_2 for α_i, v_3 for q_i and v_4 for λ_i, with:

$$\begin{cases} x &=& (L,P) \in K \\ u &\in& U = [-\frac{\Delta}{2}, \Delta] \\ v &\in& V = [2.2, 2.3] \times [0,1] \times [2.2, 2.3] \times [1/19, 1/18] \end{cases} \tag{12}$$

$$Bv \begin{cases} (L,P)'(t) &=& f_B((L,P)(t), u(t), v(t)) \\ u(t) &\in& U \\ v(t) &\in& V \\ (L,P)(t) &\in& K \end{cases} \tag{13}$$

Since U and V are constant functions of (L,P), and since f_B is Lipschitz, the conditions of Proposition 5 are fulfilled. Since U is constant, it is Lipschitz for every $\lambda > 0$, hence the guaranteed viability kernel $Guar_{\lambda, f_B}(K)$ associated with problem (13) is closed and has the property of guaranteed viability. Thus from Theorem 1 it is a consensus set of states for the four members of the committee whose beliefs are summarized in Table 2. To compute an approximation of $Guar_{\lambda, f_B}(K)$, we use the ViabLab library (Désilles, 2020), developed by A. Désilles and used in Durand et al. (2017). This library uses the convergence conditions of the algorithm established by P. Saint-Pierre (Saint-Pierre, 1994). Since the ViabLab library currently requires discrete problems in time and space for the computation of guaranteed viability, we defined a discretized version of the viability problem, with function f_{Bd} from Eq. (14), with a discretization parameter $\tau = 0.1$ for which the dynamics are stable. We also used projection on the grid method from Lavallée (2020) to minimize discretization error.

$$f_{Bd}((L,P),u,v) = \begin{pmatrix} L + \tau u \\ P + \tau \left[-v_1 P + L + \right. \\ \left. r\left((1-v_2)\frac{P^{v_3}}{m^{v_3}+P^{v_3}} + v_2 \frac{P}{x_2 + m e^{(-v_4(P-m))}}\right)\right] \end{pmatrix} \tag{14}$$

The resulting approximation $Guar_{f_{Bd}}(K)$ is shown in Fig. 5(d). The guaranteed viability kernel for the group is a viable set for all members, and the viable controls on its boundary are viable controls for all members.

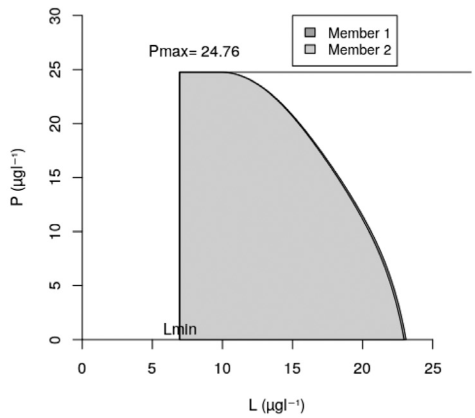
4.3. Discussion

Depending on the dynamics and the beliefs of the different group members, the guaranteed viability kernel computed following the approach in Section 3.1 could be smaller than the one corresponding to the union of the parameters set of each group member. For instance, with $(b,r) \in \{(2.1, 100), (2.2, 80)\}$, it is possible to design a guaranteed viable set for these two values only, solving the problem with the computation of integral curves as done in Section 2.3 for the lake problem (see Fig. 4). Whereas following the method in Section 3.1, in order to respect the conditions of VT theorems and use the ViabLab library it is necessary to define a more constrained guaranteed viability problem, with $(b,r) \in [2.1, 2.2] \times [80, 100]$. However when the dimension of the state space is greater than 2, the first method is virtually impossible to implement with a generic module (since a formal approach and a specific mathematical study of the dynamics are necessary).

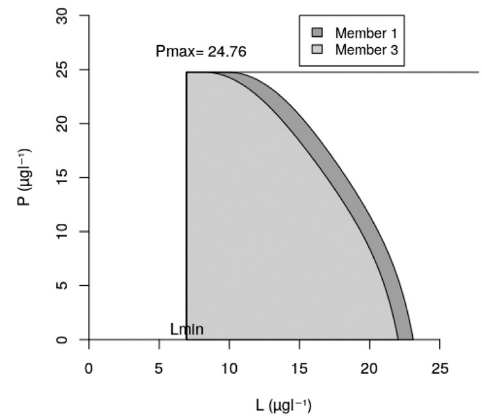
The viability algorithm is exponential with the dimension of the space in the general case, and thus it can be very slow, in particular when the viability kernel is empty or with high-dimensional problems. For each alternative model definition a tyche variable has to be considered, which increases the dimension of V linearly with the number of models. Table 3 in Appendix C shows the computation time in several cases.

When the consensus solution is the empty set, the dynamics and the admissible control map are not compatible with the set of desirable states in which the system should remain. Negotiations should take place in order to relax the constraints on the problem: by defining a bigger set of desirable states, by considering more controls, or by considering more specific dynamics. Then the guaranteed viability approach can be used again to propose a new consensus solution. This kind of stress relaxation can be seen as a problem of model exploration in the parameters space (where the parameters would be, for instance, the thresholds on the state variables, the control variables, the range of controls). There exists exploration software, such as OpenMole (Reuillon et al., 2013), that could be used to assist stakeholders during this stage.

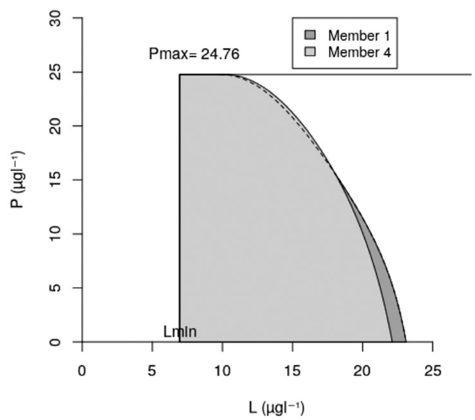
Regarding the model of Lake Bourget itself, we consider here a single control for different practices (use of wetlands, use of retention basins, different farmer practices), and the value of its range is consistent with observations but arbitrary. The model could be improved by taking into account more detailed mechanisms for these different types of control and their relation to soil leaching and rainfall. The lake is considered as homogeneous, which seams a reasonable assumption



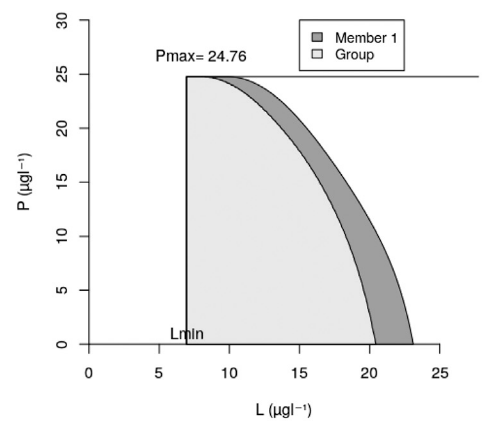
(a) Member 2 versus Member 1.



(b) Member 3 versus Member 1.



(c) Member 4 guaranteed viability kernel versus Member 1.



(d) Guaranteed Viability kernel $Guar_{f_{Bd}}$ associated with the group versus Member 1.

Fig. 5. Guarantied viability kernel for each member and the group versus the viability kernel of member 1. Computation with R (Team, 2010) and ViabLab (Désilles, 2020). Code available at https://forgemia.inra.fr/isabelle.alvarez/emlake.

regarding the water resident time of 14 years (Brias et al., 2018). On the other hand, since cyanobacteria blooms are often localized, it could be helpful to use spatial and weekly data to assess the size of perturbations and take them into account to consider robustness issues as defined in Martin and Alvarez (2019).

5. Conclusion

The viability approach (also called "co-viability", depending on authors; see, for instance, Mouysset et al. (2014)) studies sustainable management problems when constraints of different nature (economic, social, ecological, ethical, etc.) must be taken into consideration and verified simultaneously. Guaranteed viability also addresses these issues, with a worst-case approach regarding uncertainty. But it also makes it possible to deal with different models when stakeholders do not agree on the description of the dynamics of the system. In this paper, we have presented a method for reaching a viability-based consensus on the management of commons with multiple usages. We have proposed a definition of the management project and stakeholder viewpoints as well as a viability-based definition for the consensus solution as a viable set for all stakeholders with conditions on their regulation map. We have defined embedding functions that allow us to compute a guaranteed viability kernel for the associated dynamics. We have then shown that this guaranteed viability kernel is a consensus solution (Theorem 1). It can then be computed with algorithms used for viability kernel approximation. We subsequently applied this method to a management scenario for Lake Bourget. The main interest of this method is that stakeholders can retain their vision of the dynamics. Negotiations can focus on the definition of desirable states and admissible actions. This method also prepares the way for an alternative to agent-based modeling when dealing with stakeholders for the management of commons. In fact, the viability approach requires that the dynamics are represented by a set of differential equations or inclusions (difference equations in the discrete case). However, this kind of representation is not appropriate when a consensual model is sought among stakeholders, contrary to agent-based modeling. If this consensus is no longer seen as a priority, it could be possible in the future to assist stakeholders to produce models more suited to the use of MVT, based on some model databases or dynamical behavior templates. Guarantied viability can then be used to propose management solutions despite uncertainty and diversity of perceptions and objectives among stakeholders, avoiding the risk of implicit average which can occur when building a consensual model.

CRediT authorship contribution statement

Isabelle Alvarez: Conceptualization, Methodology, Software, Validation, Writing – original draft, Writing – review, Supervision. **Laetitia Zaleski:** Investigation, Conceptualization, Methodology, Software, Writing – original draft, Writing – review, Supervision. **Jean-Pierre Briot:** Conceptualization, Writing – review, Funding acquisition, Supervision. **Marta de A. Irving:** Conceptualization, Validation, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Code Data is shared here: https://forgemia.inra.fr/isabelle.alvarez/emlake and it is mentionned in the article.

Acknowledgments

We thank Sophie Martin for her help with the revised version of this article.

Appendix A. Properties of viability kernels

A.1. Properties of multi-valued maps

Let G be a multi-valued map from $\mathbb{R}^n$ to $\mathbb{R}^p$. The domain of G is $Dom(G) = \left\{x \in \mathbb{R}^n, G(x) \neq \emptyset\right\}$. The graph of G is $Graph(G) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p, y \in G(x)\}$. G has a linear growth if there is $c > 0$ such that for all $x \in Dom(G)$, $\|G(x)\| \leq c(\|x\| + 1)$. The system $Sc(f, U)$ (1) is Marchaud if f is continuous, $Graph(U)$ is closed, f and U have linear growth and the image set $\{f(x, u), u \in U(x)\}$ is convex for all of $x \in Dom(U)$. G is Lipschitz for constant $\lambda > 0$, if for all x_1, x_2 in $\mathbb{R}^n$, $G(x_1) \subset G(x_2) + \lambda\|x_1 - x_2\|B$, where B is the unit ball.

A.2. Closed Viability Kernels (Aubin, 1991)

Proposition 3 (*Continuous case*)**.** *When the system $Sc(f, U)$ (1) is Marchaud and K is closed, the associated viability kernel is closed. It is the largest viable set in K.*

Proposition 4 (*Discrete case*)**.** *When system $Sd(f, U)$ (2) is such that f is continuous, U has a linear growth, $Graph(U)$ is closed, and when K is closed, the associated viability kernel is closed.*

A.3. Guaranteed viability kernels

Definition 14. A dynamical controlled tychastic system $Svc(f, U, V)$ (4) is Lipschitz if f is Lipschitz, and U and V are Lipschitz with compact images.

Proposition 5. *From Doyen (2000) in the continuous case and Lavallée (2020) in the discrete case. The guaranteed viability kernel associated with a set K is closed when the dynamics verify the following conditions: in the continuous case, when K is closed and $Svc(f, U, V)$ is Lipschitz; in the discrete case, when f and V are continuous, $Graph(U)$ is closed and U has a linear growth (see 2.2 for the definitions).*

We recall Theorem 1: The guaranteed viability kernel associated with K for $Svc(f, U, V)$ (with λ Lipschitz constant in the continuous case) (resp. $Svd(f, U, V)$ in the discrete case) is a consensus solution to the management of $\mathcal{A}$.

Table 3
Computation time of the guaranteed viability kernel with a processor i7-8650U CPU @ 1.90 GHz × 8 and 15.5GiB RAM for different problems. Increasing the dimension of V can lead to much longer computation time. The v_{dim} column indicates the number of discretization steps used for the model type when appropriate.

State space		Control		Tyche (V)			Scenario	calculation
dim.	Steps (pts/axis)	dim.	Steps	dim.	Steps	v_{dim}		time (in s.)
Lake Bourget model (14)							P_{max}	
2	1000	1	11	3	11	5	24.76	938.64
2	1000	1	11	3	11	5	15.0	1243.86
2	1000	1	11	4	11	5	15.0 $m \in [26.0, 27.0]$	7122.99[a]
Marine-protected area (Zaleski, 2020)								
3	100	2	11	2	11	–	–	437.94
3	300	2	11	2	5	–	–	3153.31

[a]Denotes an empty guaranteed viability kernel.

Proof of Theorem 1. Let $L \neq \emptyset$ be the guaranteed viability kernel for system (4) with λ Lipschitz constant (resp. system (5) in the discrete case) associated with constraint set K. Let $\tilde{U}$ be the guaranteed regulation map. By definition, $Dom\tilde{U} = L$. We now prove that the guaranteed viability kernel is a viable set for each member i. Let $i \in \mathcal{N}$, we consider the system $Sc'_i = Sc(f_i, \tilde{U})$ (resp. $Sd'_i = Sd(f_i, \tilde{U})$ in the discrete case). Since the guaranteed viability kernel is defined from the group project (K, U), we have $L \subset K$, and for all $x \in L$, $\tilde{U}(x) \subset U(x)$. Thus an evolution governed by Sc'_i (resp. Sd'_i) is also an evolution governed by $Sc(f_i, U)$ (resp. $Sd(f_i, U)$). Since system (4) (resp. system (5)) embeds $Sc(f_i, U)$ (resp. $Sd(f_i, U)$), it also embeds Sc'_i (resp. Sd'_i). Let $x_0 \in L$, and let $x(.)$ (resp. x^k) be a trajectory starting at x_0 and governed by Sc'_i (resp. Sd'_i). Because of the embedding there is a function v such that $f_i(x(t), \tilde{u}(t)) = f(x(t), \tilde{u}(t), v(t))$ (resp. $f_i(x^k), \tilde{u}^k = f(x^k, \tilde{u}^k, v^k)$ in the discrete case). Hence $x(.)$ is also an evolution governed by system (4) (resp. (5)). Since L is the guaranteed viability kernel for system (4) with λ Lipschitz constant (resp. (5)) associated with constraint K, from Definition 12, all trajectories starting from $x_0 \in L$ and governed by (4) (resp. (5)) are viable in L for control selection in $\tilde{U}$. Thus the trajectory of $x(.)$ governed by Sc'_i (resp. Sd'_i) starting at x_0 stays in L. Thus $x(.)$ is an evolution starting at x_0 governed by Sc_i (resp. Sd_i) viable in L. Thus L is a viable set for member i. ■

Appendix B. Embedding system

Proposition 6. *If U has a linear growth and for all $i \in \mathcal{N}$, f_i has a linear growth, then a system (4) (resp. (5)) can embed $Sc(f_i, U)$ (1) (resp. $Sd(f_i, U)$ (2)) for all $i \in \mathcal{N}$.*

Proof. We note $M_x = \max_{i \in \mathcal{N}, u \in U(x)}(\|f_1(x, u) - f_i(x, u)\|)$. M_x is defined since U and all f_i have a linear growth. We define $f(x, u, v) = f_1(x, u) + v(x)$ with $v(x) \in V(x) = B(0, M_x)$, where $B(a, r)$ is the closed ball with center a and radius r. Then for $x \in K$ and $u \in U(x)$ we define $v_{i,u,x} = f_i(x, u) - f_1(x, u)$. We have $\|v_{i,u,x}\| \leq M_x$, and thus $v_{i,u,x} \in V(x)$. Then $\forall i \in \mathcal{N}, f_i(x, u) = f(x, u, v_{i,u,x})$ and Eq. (6) is verified. ■

Definitions of f leading to smaller sets of perturbation are preferable. For instance, it can be interesting to define f with the convex hull of the $f_i(x, u)$: With $i \in J = \mathcal{N} \setminus \{1\}$ we consider $v = (v_i), v_i \in [0, 1]$ with $\sum_{i \in J} v_i \leq 1$, and define $f(x, u, v) = f_1(x, u)(1 - \sum_{i \in J} v_i) + \sum_{i \in J} v_i f_i(x, u)$.

Appendix C. Computing performance for guarantied viability

See Table 3.

References

Alais, J.-C., Carpentier, P., De Lara, M., 2017. Multi-usage hydropower single dam management: chance-constrained optimization and stochastic viability. Energy Syst. 8 (1), 7–30.

Aubin, J.-P., 1991. Viability Theory. Birkhäuser, Basel.

Aubin, J.-P., 1997. Dynamic Economic Theory: A Viability Approach, Vol. 5. Springer Verlag.

Barreteau, O., Bousquet, F., Attonaty, J.-M., 2001. Role-playing games for opening the black box of multi-agent systems: method and lessons of its application to Senegal River. J. Artif. Soc. Soc. Simul. 4 (2), 5.

Bates, S., Saint-Pierre, P., 2018. Adaptive policy framework through the lens of the viability theory: A theoretical contribution to sustainability in the anthropocene era. Ecol. Econom. 145, 244–262.

Brias, A., Mathias, J.-D., Deffuant, G., 2018. Inter-annual rainfall variability may foster lake regime shifts: An example from Lake Bourget in France. Ecol. Model. 389, 11–18.

Carpenter, S.R., Ludwig, D., Brock, W.A., 1999. Management of Eutrophication for Lakes Subject to Potentially Irreversible Change. Ecol. Appl. 9 (3), 751–771.

de Groot, R., 2006. Function-analysis and valuation as a tool to assess land use conflicts in planning for sustainable, multi-functional landscapes. Landsc. Urban Plan. 75 (3), 175–186.

De Lara, M., Martinet, V., 2009. Multi-criteria dynamic decision under uncertainty: A stochastic viability analysis and an application to sustainable fishery management. Math. Biosci. 217 (2), 118–124.

Désilles, A., 2020. ViabLab Library. Technical report, LASTRE, https://github.com/lastre-viab/VIABLAB.

Dowling, N.A., Dichmont, C.M., Leigh, G.M., Pascoe, S., Pears, R.J., Roberts, T., Breen, S., Cannard, T., Mamula, A., Mangel, M., 2020. Optimising harvest strategies over multiple objectives and stakeholder preferences. Ecol. Model. 435, 109243.

Doyen, L., 2000. Lipschitz Kernel of a Closed Set-Valued Map. Set-Valued Anal. 8 (1), 101–109.

Durand, M.-H., Désilles, A., Saint-Pierre, P., Angeon, V., Ozier-Lafontaine, H., 2017. Agroecological transition: A viability model to assess soil restoration. Nat. Resour. Modeling 30 (3), e12134.

Etienne, M. (Ed.), 2014. Companion Modelling: A Participatory Approach to Support Sustainable Development. Springer Netherlands.

Gajardo, P., Harmand, J., Ramirez, H., Rapaport, A., Riquelme, V., Rousseau, A., 2017. Modeling and control of in-situ decontamination of large water resources. In: Serea, O.S., Briec, W. (Eds.), ESAIM: Proc. Surv. 57, 70–85.

Hardin, G., 1968. The Tragedy of the Commons. Sci., New Ser. 162 (3859), 1243–1248.

Jacquet, S., 2018. Suivi scientifique du lac du Bourget pour l'année 2017. Technical report, INRAE ; CISALB ; Agence de l'eau RMC.

Křivan, V., 1991. Differential inclusions as a methodology tool in population biology. In: Proceedings of the 1995 European Simulation Multiconference. Prague, pp. 544–547.

Křivan, V., Colombo, G., 1998. A Non-stochastic Approach for Modeling Uncertainty in Population Dynamics. Bull. Math. Biol. 60 (4), 721–751.

Lavallée, F., 2020. Model, Exploration, Viability Theory for Management Support of Invasive Species (PhD thesis). Sorbonne University, In French, https://tel.archives-ouvertes.fr/tel-03836187.

Little, L.R., Punt, A.E., Mapstone, B.D., Pantus, F., Smith, A.D.M., Davies, C.R., McDonald, A.D., 2007. ELFSim—A model for evaluating management options for spatially structured reef fish populations. Ecol. Model. 205 (3), 381–396.

Liu, W., Wang, S., Zhang, L., Ni, Z., 2015. Water pollution characteristics of Dianchi Lake and the course of protection and pollution management. Environ. Earth Sci. 74 (5), 3767–3780.

Martin, S., 2004. The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication. Ecol. Soc. 2 (9), 8.

Martin, S., Alvarez, I., 2019. Anticipating shocks in the state space: Characterizing robustness and building increasingly robust evolutions. SIAM J. Control Optim. 57 (1), 490–509.

Martinet, V., Peña-Torres, J., De Lara, M., Ramírez C., H., 2016. Risk and Sustainability: Assessing Fishery Management Strategies. Environ. Resour. Econ. 64 (4), 683–707.

Mouysset, L., Doyen, L., Jiguet, F., 2014. From Population Viability Analysis to Coviability of Farmland Biodiversity and Agriculture. Conserv. Biol. 28 (1), 187–201.

Oubraham, A., Zaccour, G., 2018. A Survey of Applications of Viability Theory to the Sustainable Exploitation of Renewable Resources. Ecol. Econom. 145, 346–367.

Parrachino, I., Dinar, A., Patrone, F., 2006. Cooperative Game Theory and Its Application To Natural, Environmental, and Water Resource Issues : 3. Application To Water Resources. Policy Research Working Papers, The World Bank.

Reuillon, R., Leclaire, M., Rey-Coyrehourcq, S., 2013. OpenMOLE, a workflow engine specifically tailored for the distributed exploration of simulation models. Future Gener. Comput. Syst. 29 (8), 1981–1990.

Saint-Pierre, P., 1994. Approximation of the viability kernel. Appl. Math. Optim. 29 (2), 187–209.

Sanogo, C., Ben Miled, S., Raissi, N., 2012. Viability Analysis of Multi-fishery. Acta Biotheor. 60 (1), 189–207.

Team, R.D.C., 2010. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna.

Vinçon-Leite, B., Tassin, B., 1990. Modélisation de la qualité des eaux des lacs profonds : modèle thermique et biogéochimique du lac du Bourget. La Houille Blanche (3–4), 231–236.

Vollenweider, 1982. coord. Eutrophication of waters. Monitoring, Assessment and Control. Technical report, OECD, p. 164.

Wei, W., Alvarez, I., Martin, S., 2013. Sustainability analysis: Viability concepts to consider transient and asymptotical dynamics in socio-ecological tourism-based systems. Ecol. Model. 251, 103–113.

Zaleski, L., 2020. Decision and Negociation Assistant by Viability Analysis: Application to the Participatory Management of Protected Areas (PhD thesis). Sorbonne University, In French, https://hal.archives-ouvertes.fr/tel-03768905v1.